Exam algebraic topology October 2024

Always explain your answers. It is allowed to refer to definitions, lemmas and theorems from the lecture notes but not to other sources. All questions are independent and count equally so make sure you try each of them. Good luck!

1. Prove that $\langle\{\{a,b\}\}\rangle \approx \langle\{\{a,c\}\}\rangle$ by providing the necessary sequence of collapses/expansions.

2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = (x^3y^2 + xy, x^2 - y^3)$. Is f homotopic to the identity $\mathrm{id}_{\mathbb{R}^2}$?

3. Suppose K is a finite abstract simplicial complex equivalent to a Klein bottle surface. Does the simplicial map $f : \mathbb{S}^2 \times K \to K$ defined by $f(x,y) = y$ for all $x \in V(\mathbb{S}^2)$ and $y \in V(K)$ define a regular covering space of K?

4. In the barycentric subdivision $(\mathbb{S}^3)'$ of the 3-sphere choose 3-simplices σ, τ such that $\langle\sigma\rangle \cap \langle\tau\rangle = \emptyset$ and define $\Phi = (\mathbb{S}^3)' \setminus \{\sigma, \tau\}$. Show that there exists an exact sequence

$$\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z} \to H_2(\Phi) \to 0$$

5. Is it possible to have two finite abstract simplicial complexes Σ and Φ and a point $b \in V(\Sigma) \cap V(\Phi)$ such that both $\pi_1(\Sigma, b)$ and $\pi_1(\Phi, b)$ and $\pi_1(\Sigma \cap \Phi, b)$ are the trivial group but $\pi_1(\Sigma \cup \Phi, b)$ is non-trivial? If yes give an explicit example, if no give a proof that this is impossible.

6. Explicitly write down a matrix for the linear map $\partial_2 : C_2(\Sigma; \mathbb{Q}) \to C_1(\Sigma; \mathbb{Q})$ where $\Sigma = \langle\{\{0,1,2,3\}\}\rangle = \mathbb{D}^3$.

7. Explain why there is only one connected covering space $f : \Sigma \to \mathbb{S}^2$ up to isomorphism.